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Transport and Diffusion in a Random Medium

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We consider particle transport in a spatially random medium, the transport governed by the traditional, linear, time- and space-dependent transport equation for "host and guest." The scattering is elastic and isotropic; there is no absorption. If the host medium has uniform density we know that an initial burst will, in time, approach the solution to the time-dependent diffusion equation. In the case of random medium we find that for a large class of such media the asymptotic behavior is unchanged by the stochasticity; there is neither renormalization of the equation nor the diffusion co-efficient.The nature of the correlation between fluctuations of density at large separation plays an important role in the analysis.

KEY WORDS: Transport equation; random medium; host-and-guest.

1. INTRODUCTION

This essay is concerned with particle transport and diffusion in a loss-less random medium, transport being described by the "one-speed transport equation" with isotropic scattering. The host medium is random in that its local density is described by a probability distribution that is stationary. There is no flow of material. One might imagine an experiment in which particles are released at the origin in a burst and, encountering an irregular, but static distribution of scatterers, diffuse to infinity. The experiment is repeated, but now the distribution of scatterers are different ... and so on. We are interested in averages with respect to the ensemble of distributions encountered and, in particular, in the "asymptotic" (distant in space and time) behavior of the averaged particle distribution. Does it obey the diffusion equation? If so, with what diffusion coefficient? Our most interesting conclusion is that for a large class of densities not only does the

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particle distribution satisfy the diffusion equation, but the coefficient is that appropriate to the mean density of the scatterers. Fluctuations about the mean have no effect upon the asymptotic behavior; there is no "renormalization." We point out that should one replace the transport equation by a different model – a diffusion equation with stochastic diffusion coefficient – the results are quite different.

The subject of transport in stochastic media has received considerable attention in the last several decades. Work prior to 1991 is described in Pomraning's pioneering monograph.(1) Less accessible is the important monograph by Stepanov.(2) For recent activity one might consult the work of Akcasu and Williams,(3) Larsen,(4) Prinja,(5) and Williams.(6) And the interested reader might consult the earlier, impressive reviews by van Beijeren(7) and by Spohn(8). In these the authors point to results similar to ours, for different Lorentz models. For wave propagation see Frisch.(16)

2. ANALYSIS

To quote J.B. Keller,(9) "A random medium is a family of media, each labeled by one value of $\alpha\ldots$ a parameter which ranges over a space A in which a probability density $p(\alpha)$ is defined. The probability density $p(\alpha)$ determines the probability of a given value of α and therefore of the corresponding (transport) equation of the family." We will write the host density as

$$n(x,\alpha) = n_0\hat{n}(x,\alpha) = n_0(1+\epsilon\,\theta(x,\alpha)). \tag{1}$$

$$\int_{-\infty}^{\infty} d\alpha\, p(\alpha)\theta(x,\alpha) = 0,$$

and treat only variations and solutions which depend upon a single spatial variable. We also do not pay much attention to details of $p(\alpha)$ and $\theta(x,\alpha)$. Concern about $n(x,\alpha)$ assuming non-physical – i.e., negative-values is responded to in the final condition that our results be "physical." We seek the particle distribution evolving, in time, from an initial burst. Most interesting to us is the asymptotic diffusion of the particles, which is expected to follow the traditional time-dependent diffusion equation. To help us we will re-write the transport equation as a "generalized diffusion equation" for the angle-integrated distribution. Thus we will be using the language and technique of the "projection operator formalism."(10,11) After a limiting process, the "asymptotic" equation will be characterized by a (renormalized?) diffusion co-efficient $D(\epsilon,\ldots)$, and we ask: How does the final diffusion reflect, or capture, the nature and degree of the randomness? We shall see that, in fact, for a large family of fluctuations the diffusion constant is unchanged "to all orders of ϵ."

We limit ourselves to distributions which depend only upon one spatial variable, and one velocity variable $v_x = v\mu$. The scattering is elastic and isotropic. In terms of dimension-less variables $x = \frac{x'}{\ell}$, $t = \frac{v}{\ell}t'$, where the primed variables are the "physical" ones and v, ℓ are the speed and mean-free-path, we have

$$\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial \mathrm{x}} + \hat{n}(x,\alpha)\ \ F(x,\mu,t,\alpha) = \hat{n}(x,\alpha) \int_{-1}^{1} \frac{d\mu}{2} F(x,\mu,t,\alpha),$$

or

$$\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial \mathrm{x}} + \hat{n}(x,\alpha)\ \ QF(x,\mu,t,\alpha) = 0, \tag{2}$$

where, for convenience, we have introduced the projection operators, $P = \frac{1}{2}\int_{-1}^{1} d\mu$ and $Q = 1 - P$, $PQ = QP = 0$. The mean free path is ℓ, is $= \frac{1}{n_0\sigma}$, and the densities will be sampled from some universe of acceptable functions, labeled by α. One should note that while the length and time scales we have introduced appear to be "natural" to the transport equation, variations in the perturbed density $\theta(x)$ may be expected to introduce another scale of variation. (Of course, in the static problem the varying density may be removed in favor of an "optical depth," the subsequent analysis being quite different.) Finally, the equation we have written is source-less; we are thinking in terms of an initial-value-problem, an initial "burst" whose angular distribution is isotropic. And we do not adhere to the structure of rigorous mathematical proof; we merely compute and find a noteworthy result.

2.1. Preliminary Calculation

Since we will be using the calculus of projection operators throughout, it may be useful to begin with an instructive example, the construction of a generalized diffusion equation for the density $F_0(x,t,\alpha) \equiv P\ \ F(x,\mu,t,\alpha)$. We write Eq. (2) as

$$\begin{gathered}\tfrac{\partial F}{\partial t} + L(\alpha)F(x,\mu,t,\alpha) = 0,\\ L(\alpha)F(x,\mu,t,\alpha) \equiv \mu\tfrac{\partial F}{\partial x} + \hat{n}(x,\alpha)QF(x,\mu,t,\alpha),\\ L_0F(x,\mu,t,\alpha) \equiv \mu\tfrac{\partial F}{\partial x} + QF(x,\mu,t,\alpha)\end{gathered} \tag{3}$$

We begin with Eq. (3), operating on it with P and Q in turn, then eliminating QF, to obtain the formal expression

$$\left(\frac{\partial}{\partial t}+PL\right)PF-PL\frac{1}{\frac{\partial}{\partial t}+Q\mathrm{L}}QL\ \ PF=0. \tag{4}$$

Now if one notes that

$$PL\ \ P()=0, \quad PL\ \ ()=P\mu\frac{\partial}{\partial x}()=\frac{\partial}{\partial x}P\mu(), \quad LP()=\mu\frac{\partial}{\partial x}()$$

one obtains the generalized diffusion equation,

$$\frac{\partial}{\partial t}F_0(x,t,\alpha)-\frac{\partial}{\partial x}P\left\{\mu\frac{1}{\frac{\partial}{\partial t}+QL(\alpha)}\mu\right\}\frac{\partial}{\partial x}F_0(x,t,\alpha)=0 \tag{5}$$

the quantity $P\{\ldots\}$ being a linear operator, $\mathbb{D}_\alpha$, operating on the x and t variables and whose features need discussion. (We will drop the label α, unless it is relevant to the discussion.) The "diffusion" is non-local in space and time. Eq. (5) is entirely equivalent to the time-dependent equation of traditional transport theory.[12]

$\mathbb{D}_\alpha$ may be calculated easily in the case of uniform density. (See Appendix A.) Then, the usual Fourier–Laplace transformation puts Eq. (5) into the form

$$[s+\widetilde{D}(k,s)k^2]\ \ \widetilde{F}_0(k,s)=1, \tag{6}$$

in the case of an initial, isotropic burst of unit strength centered at $x=0$. Here $\widetilde{F}_0(k,s)$ is the transform of $F_0(x,t)$ and

$$\mathbb{D}\exp(st+ikx)=\widetilde{D}(k,s)\exp(st+ikx).$$

In this case we find

$$\widetilde{D}(k,s)=\left\langle\frac{\mu^2}{d}\right\rangle-\frac{\left\langle\frac{\mu}{d}\right\rangle^2}{\left\langle\frac{1}{d}\right\rangle}. \tag{7}$$

$$=\Lambda_2-\frac{\Lambda_1^2}{\Lambda}. \tag{7a}$$

In this tidy expression P has been replaced by the "averaging symbol" $\langle \dots \rangle$, a change in notation which, with apologies to the reader, we make occasionally in the interest of clarity. Further, $d = d(k, \mu, s) = 1 + s + ik\mu$ and the Λ-symbols, which will prove helpful, are functions of (k, s), defined through

$$\Lambda_{\mathrm{m}}(k, s) = \left\langle \frac{\mu^{\mathrm{m}}}{d} \right\rangle.$$

Simple relations, such as

$$\begin{aligned} (1+s)\Lambda_1 + ik\Lambda_2 &= 0 \\ ik\Lambda_1 + (1+s)\Lambda &= 1 \end{aligned} \tag{8a}$$

connect the symbols; then $\widetilde{D}(k, s) = \Lambda_2 - \frac{\Lambda_1^2}{\Lambda} = \frac{i}{k}\frac{\Lambda_1}{\Lambda}$, and we will encounter the null combination,

$$\Lambda_2 + \frac{\Lambda_1^2}{1-\Lambda} = 0 \quad (s = 0) \tag{8b}$$

Equation (6) implies that Eq. (5) may also be written, with a non-local kernel, D, as

$$\frac{\partial}{\partial t} F_0(x, t) - \frac{\partial}{\partial x} \int_{-\infty}^{\infty} dx' \int_0^t dt' D(x - x', t - t') \frac{\partial}{\partial x'} F_0(x', t') = 0, \tag{9}$$

and that fact brings us to the "Markoffian Limit" (or approximation).

The evolution of the initial burst is complicated. The distribution has both causal – it has a "front" – and diffusive behavior behind the front. Two time and space-scales are present, the mean-free-time and space scales for collision, which are short, and the long time and space scales for the diffusive evolution of the distribution. Roughly, they are expressed in the mathematics through contributions from a discrete spectrum (poles) and a continuous spectrum (branch cuts), the former describing long-time behavior. More precisely, consider behavior for complex-s and fixed, real k. Then, a pole, $s(k)$, will describe a "branch," or generate a dispersion relation for bulk motion.[13] The accompanying continuum contribution will describe transient, "high-frequency" behavior. One notices (see Appendix A) that while $\widetilde{F}_0(k, s)$ has poles, $\widetilde{D}(k, s)$ does not. We use this observation-along with "physical intuition" to assert that the diffusion kernel $D(x, t)$ in Eq. (9) indeed relaxes to zero with increase of its arguments at a rate much faster than the relaxation of $F_0(x,t)$. That is, the diffusion kernel quickly becomes "local." Carrying this notion to its limit produces the Markoffian approximation, which may be expressed concisely by writing

$$D(x-x',t-t') = \widetilde{D}(k=0,\ s=0)\delta(x-x')\delta(t-t'). \tag{10}$$

In that limit $d=1$, $\widetilde{D}(0,0)=\frac{1}{3}$, and we retrieve the traditional

$$\frac{\partial}{\partial t}F_0(x,t) - \frac{1}{3}\frac{\partial^2}{\partial x^2}F_0(x,t) = 0 \tag{11}$$

The process leading from Eq. (9) to Eq. (11) connects – as it were – the mesoscopic with the macroscopic world.

A final issue concerns the manner in which the limits $k\to 0$, $s\to 0$ are carried out. While it is true that in some cases, e.g. the Telegrapher's equation, whose propagator is $(s^2+s+k^2)^{-1}$, results will depend upon the process, we will not find it to be the case, here

2.2. Averaging the Irregularities

We wish to discuss the effects of two averagings, or projections. Along with angle-averaging, noted above, we consider averaging over the solutions with $p(\alpha)$,

$$\int d\alpha\ \ p(\alpha) = 1$$

and introduce additional projection operators P_α, $Q_\alpha = (1-P_\alpha)$,

$$P_\alpha\, F(\ldots,\alpha) = \int d\alpha\, p(\alpha) F(\ldots,\alpha).$$

P_α, Q_α commute with P, Q, and $\mathcal{D}$, We also have $P_\alpha \hat{n}(x,\alpha) = 1$, so that

$$Q_\alpha \hat{n}(x,\alpha) = \epsilon\,\theta(x,\alpha).$$

Now we may subject the transport equation, Eq. 3, to the same operations we sketched above, (see Appendix B) operating, in turn, with P_α and Q_α, then eliminating $Q_\alpha F$, to get the equation

$$\left(\frac{\partial}{\partial t} + L_0\right)\langle F\rangle_\alpha(x,\mu,t) - \epsilon^2 Q\Sigma(\epsilon)Q\langle F\rangle_\alpha(x,\mu,t) = 0, \tag{12}$$

for the averaged (projected) distribution. It replaces the original equation

$$\left(\frac{\partial}{\partial t}+L_0\right)F(x,\mu,t,\alpha)+\epsilon\theta(x,\alpha)QF(x,\mu,t,\alpha)=0. \tag{13}$$

The operator $\Sigma(\epsilon)$ in Eq. (12) is the complicated object

$$\begin{aligned}\Sigma(\epsilon) &= P_\alpha\left[\theta(x,\alpha)\frac{1}{\frac{\partial}{\partial t}+L_0+\epsilon QQ_\alpha\theta(x,\alpha)}\theta(x,\alpha)\right]\\ &= P_\alpha[\theta(x,\alpha)\mathcal{G}\ (\epsilon,\alpha)\theta(x,\alpha)]\end{aligned} \tag{14}$$

an operator which acts upon functions of (x,μ,t). Equation (12) in one form or another may be found throughout the literature – perhaps as "Dyson's Equation" in its earliest appearance.[14]

We complete the general discussion first by improving notation, writing

$$\langle F\rangle_\alpha(x,\mu,t)\equiv\psi(x,\mu,t),$$

then using the angle-averaging operators to reduce Eq. (12) to a generalized diffusion equation – in the presence of a "random medium." We find

$$\frac{\partial}{\partial t}\psi_0(x,t)-\frac{\partial}{\partial x}P\left\{\mu\frac{1}{\frac{\partial}{\partial t}+Q(L_0-\epsilon^2\Sigma(\epsilon))}\mu\right\}\frac{\partial}{\partial x}\psi_0(x,t)=0, \tag{15}$$

an equation which, while compact, is quite complicated. Nevertheless, we proceed just as we did in the simple case of Eq. (5). We have a different diffusion operator $\mathbb{D}(\epsilon)$, but our assumption of stationary and homogeneous statistics ensures that again

$$\mathbb{D}(\epsilon)\exp(st+ikx)=\widetilde{D}(k,s,\epsilon)\exp(st+ikx),$$

and *if* a Markoffian limit of the operator-kernel $\widetilde{D}(k,s,\epsilon)$ exists, it will provide a "renormalized" diffusion coefficient. The calculations are tedious but the strategy should be clear.

2.3. ϵ-Expansion and Green's Functions

To make progress one usually assumes that irregularities are small – that $\epsilon \ll 1$. One is then led, naturally, to the expansion of

$$\mathcal{G}_1(\epsilon) \equiv \frac{1}{\frac{\partial}{\partial t} + Q(L_0 - \epsilon^2 \Sigma)}, \quad \mathcal{G}(\epsilon, \alpha) = \frac{1}{\frac{\partial}{\partial t} + \mathrm{L}_0 + \epsilon\, Q\, Q_\alpha \theta(x, \alpha)}$$

where

$$\mathbb{D}(\epsilon) \equiv P\{\mu \mathcal{G}_1(\epsilon)\, \mu\}, \quad \Sigma(\epsilon) = P_\alpha[\theta(x, \alpha)\, \mathcal{G}(\epsilon, \alpha) \theta(x, \alpha)]$$

We find

$$\mathcal{G}_1(\epsilon) = G_1 + \epsilon^2 G_1 Q\, \Sigma(\epsilon) G_1 + \epsilon^4 G_1 Q\, \Sigma(\epsilon) G_1 Q\, \Sigma(\epsilon) G_1 + \dots \tag{16}$$

with the inverse operators $G = \frac{1}{\frac{\partial}{\partial t} + L_0}$ and $G_1 = \frac{1}{\frac{\partial}{\partial t} + \mathrm{Q} L_0}$.
As Green's functions, G and G_1 obey

$$\left(\frac{\partial}{\partial t} + L_0\right) G(x - x', t - t', \mu, \mu') = \delta(x - x') \delta(t - t')\, \delta(\mu - \mu') \tag{17}$$

and

$$\left(\frac{\partial}{\partial t} + Q L_0\right) G_1(x - x', t - t', \mu, \mu') = \delta(x - x') \delta(t - t')\, \delta(\mu - \mu').$$

Note that the latter has the interesting feature

$$\frac{\partial}{\partial t} P G_1(x, t, \mu, \mu') = \frac{1}{2} \delta(x) \delta(t),$$

with the particular solution

$$(P G_1)(x, t) = \frac{1}{2} \delta(x) H(t).$$

It is natural to conduct our analysis in terms of Fourier–Laplace transforms, whereupon

$$\begin{aligned}\widetilde{G}(k, s, \mu, \mu') &= \frac{1}{2}\left[\frac{1}{1 - \Lambda(k, s)} \frac{1}{d(\mu) d(\mu')} + \delta(\mu - \mu') \left\{\frac{1}{d(\mu)} + \frac{1}{d(\mu')}\right\}\right] \\ \widetilde{G}_1(k, s, \mu, \mu') &= \frac{1}{2}\left[\frac{1}{s} \frac{1 + ik\mu'}{\Lambda(k, s)} \frac{1}{d(\mu) d(\mu')} + \delta(\mu - \mu') \left\{\frac{1}{d(\mu)} + \frac{1}{d(\mu')}\right\}\right].\end{aligned} \tag{18}$$

Since we will be representing $\psi_0(x,t)$ as Fourier–Laplace transforms, crucial to our calculation is the observation that – for arbitrary $\phi(\mu)$

$$\begin{aligned}
\frac{1}{\frac{\partial}{\partial t}+L_0}\phi(\mu)e^{ikx+st} &= e^{ikx+st}\int_{-1}^{1} d\mu' \widetilde{G}(k,s,\mu,\mu')\phi(\mu'),\\
\int_{-1}^{1} d\mu' \widetilde{G}(k,s,\mu,\mu')\phi(\mu') &= \tfrac{1}{d(k,\mu)}\left[\phi(\mu)+\tfrac{1}{1-\Lambda(k,s)}\left\langle\tfrac{\phi}{d(k,\mu)}\right\rangle\right]\\
\int_{-1}^{1} d\mu' \widetilde{G}_1(k,s,\mu,\mu')\phi(\mu') &= \tfrac{1}{d(k,\mu)}\left[\phi(\mu)+\tfrac{1}{s\,\Lambda(k,s)}\left\langle\tfrac{1+ik\mu}{d(k,\mu)}\,\phi\right\rangle\right] \qquad (19)\\
&= \tfrac{1}{d(k,\mu)}\left[\phi(\mu)+\tfrac{1}{s\,\Lambda(k,s)}\left\{\langle\phi\rangle - s\left\langle\tfrac{\phi}{d(k,\mu)}\right\rangle\right\}\right]
\end{aligned}$$

In the future we may abbreviate these equations as

$$\begin{aligned}
\widetilde{G}(k)\cdot\phi(\mu) &= \frac{1}{d(k,\mu)}\left[\phi(\mu)+\frac{1}{1-\Lambda(k)}\left\langle\tfrac{\phi}{d(k)}\right\rangle\right] \qquad (20)\\
\widetilde{G}_1(k)\cdot\phi(\mu) &= \frac{1}{d(k,\mu)}\left[\phi(\mu)+\frac{1}{s\,\Lambda(k)}\left\langle\tfrac{1+ik\mu}{d(k)}\,\phi\right\rangle\right], \text{ etc.}
\end{aligned}$$

for convenience. (The Laplace variable, s, is implicit.) Should the mean value of ϕ vanish, then $\phi(\mu)=Q\chi(\mu)$ and we have

$$\begin{aligned}
\widetilde{G}(k)\cdot Q\chi(\mu) &= \frac{1}{d(k,\mu)}\left[\chi(\mu)-\frac{1}{1-\Lambda(k)}\left\{\langle\chi\rangle-\left\langle\frac{\chi}{d(k)}\right\rangle\right\}\right] \qquad (21)\\
\widetilde{G}_1(k)\cdot Q\chi(\mu) &= \frac{1}{d(k,\mu)}\left[\chi(\mu)-\frac{1}{\Lambda(k)}\left\langle\frac{\chi}{d(k)}\right\rangle\right]
\end{aligned}$$

Also useful is:

$$P\,\mu\widetilde{G}_1(k)\cdot\phi(\mu)=\left\langle\frac{\mu\,\phi}{d(k)}\right\rangle+\frac{\Lambda_1}{s\,\Lambda}(k)\left\{\langle\phi\rangle-s\left\langle\frac{\phi}{d(k)}\right\rangle\right\} \qquad (22)$$

which, for $\phi(\mu)=Q\chi(\mu)$, becomes

$$P\,\mu\widetilde{G}_1(k)\cdot Q\chi(\mu)=\left\langle\frac{\mu}{d(k)}\chi\right\rangle-\frac{\Lambda_1}{\Lambda}(k)\left\langle\frac{1}{d(k)}\chi\right\rangle. \qquad (23)$$

The last result produces Eq. (7) immediately upon setting $\chi(\mu)=\mu$. Finally,

$$Q\,\widetilde{G}(k)\cdot\phi(\mu)=\frac{1}{d(k,\mu)}\left[\phi(\mu)\;-\frac{d-1}{1-\Lambda(k,s)}\left\langle\frac{\phi(\mu)}{d}\right\rangle\right] \qquad (24)$$

which, in the important case, $s=0$, becomes

$$Q\,\widetilde{G}(k)\cdot\phi(\mu)=\frac{1}{d(k,\mu)}\left[\,\phi(\mu)-\frac{\mu}{\Lambda_1}\left\langle\frac{\phi(\mu)}{d}\right\rangle\right]\quad (s=0).$$

In particular,

$$Q\,\widetilde{G}(k)\cdot\mu=0\quad (s=0)\qquad (*) \tag{25}$$

These are a few of the results of the "projector-calculus." Since we will be concerned with the Markoffian limit, in sequence $s\to 0$, then $k\to 0$, we will want the limiting forms of several of the above statements. These will be quite simple since then, $d\to 1$, $\Lambda\to 1$, and $\Lambda_1\to 0$. And now we are ready to consider the operator $\Sigma(\epsilon)$.

2.4. The Operator $\Sigma(\epsilon)$

An expansion in ϵ,

$$\Sigma(\epsilon)=\Sigma_0+\epsilon\,\Sigma_1+\epsilon^2\,\Sigma_2+\cdots,$$

is made via expansion of $\mathcal{G}(\epsilon,\alpha)$. Then,

$$\mathcal{G}_1(\epsilon)=G_1+\epsilon^2\,G_1Q\,\Sigma(\epsilon)G_1+\epsilon^4G_1Q\,\Sigma(\epsilon)G_1Q\,\Sigma(\epsilon)G_1+\cdots \tag{26}$$

becomes

$$\begin{aligned}\mathcal{G}_1(\epsilon)\;=\Big[&G_1+\epsilon^2\,G_1Q\,(\Sigma_0+\epsilon\,\Sigma_1+\epsilon^2\,\Sigma_2+\cdots)G_1\\ &+\epsilon^4G_1Q\,\Sigma_0G_1Q\,\Sigma_0G_1+\cdots\Big]\end{aligned}$$

through fourth order. Now,

$$\mathcal{G}(\epsilon,\alpha)=G-\epsilon\,G\,Q_\alpha\theta\,Q\,G+\epsilon^2\,GQ_\alpha\theta\,Q\,G\,Q_\alpha\theta Q\,G-\cdots \tag{27}$$

leads to

$$\begin{aligned}\Sigma_0&=\langle\theta G\,\theta\rangle_\alpha\\ \Sigma_1&=\langle\theta\,GQ_\alpha\theta\,Q\,G\theta\rangle_\alpha=\langle\theta\,G\theta\,Q\,G\theta\rangle_\alpha\end{aligned} \tag{28}$$

and

$$\Sigma_2 = \langle \theta\, G\, Q_\alpha \theta\, Q\, G Q_\alpha \theta\, Q\, G\theta \rangle_\alpha = \langle \theta\, G\, \theta\, Q\, G Q_\alpha \theta\, Q G\theta \rangle_\alpha$$
$$= \langle \theta\, G\theta\, Q\, G\theta\, Q G\theta \rangle_\alpha - \langle \theta\, G\theta\, \rangle_\alpha\, Q\, G\, Q \langle \theta\, G\theta \rangle_\alpha$$

(Note that Σ_2 suggests that we are generating a cumulant expansion rather than a traditional expansion. This is another appealing feature of the projection calculus.[15,16])

We will now evaluate the lowest order correction to the diffusion kernel in some detail. For that we need ...

2.5. The Operator Σ_0

If we use a Fourier representation for $\theta(x, \alpha)$ we find

$$\Sigma_0 = \frac{1}{(2\pi)^2} \int dk' \int dk'' e^{ik'x} \langle \widetilde{\theta}(k', \alpha) \widetilde{\theta}(k'', \alpha) \rangle_\alpha \frac{1}{\frac{\partial}{\partial t} + L_0} e^{ik''x} \tag{29}$$

We assume that the system is homogeneous, that the fluctuations are no different in one region than other. Then,

$$\langle \theta(x, \alpha) \theta(x', \alpha) \rangle_\alpha = C(x - x'), \tag{30}$$

and we may take

$$\langle \widetilde{\theta}(k', \alpha)\, \widetilde{\theta}(k'', \alpha) \rangle_\alpha = 2\pi\, \Theta(k') \delta(k' + k'').$$

$\Theta(k')$ and the correlation function $C(x)$ form a Fourier transform pair. Further, the quantity $\Theta(k')$ is related to the local "power spectrum" of the fluctuations through

$$\langle \theta^2(x, \alpha) \rangle_\alpha = \frac{1}{2\pi} \int dk' \Theta(k') = C(0).$$

(The power spectrum is independent of position.)
Equation (29) then becomes

$$\Sigma_0 = \frac{1}{2\pi} \int dk' \Theta(k') e^{ik'x} \frac{1}{\frac{\partial}{\partial t} + L_0} e^{-ik'x}, \tag{31}$$

$$\begin{aligned}\Sigma_0 \phi(\mu, k, s) e^{ikx+st} &= e^{ikx+st} \frac{1}{2\pi} \int dk' \Theta(k') G(k - k', s) \cdot \phi(\mu, k, s) \\ &= e^{ikx+st} \frac{1}{2\pi} \int dk' \Theta(k - k') G(k', s) \cdot \phi(\mu, k, s),\end{aligned} \tag{32}$$

$G\cdot$ operating only upon μ.

A word about "modeling" $C(x)$: It is natural to introduce a correlation length, ξ, and to model $C(x)$ as, say

$$C(x) = C(0)e^{-\frac{|x|}{\xi}},$$

whereupon

$$\Theta(k) = C(0)\frac{2\xi}{1+k^2\xi^2}.$$

Then, in the limit of correlation length much larger than mean-free-path ($\xi \to \infty$, homogeneous medium) we have

$$\Theta(k) \to 2\pi C(0)\delta(k), \tag{33}$$

while in the limit of vanishing length ($\xi \to 0$, white-noise, uncorrelated fluctuations) we would model

$$C(x) = \frac{C_0}{2\xi}e^{-\frac{|x|}{\xi}} \to C_0\delta(x)$$

so as to get

$$\Theta(k) \to \text{constant} = C_0 \text{ in that limit.}$$

We shall use, exclusively, models for which $\Theta(-k) = \Theta(k)$. And, as a final comment, remark that our analysis is limited to those realizations, $\theta(x,\alpha)$, and probabilities $p(\alpha)$ for which expressions like Eq. (32) and Eq. (34) (ahead) are meaningful.

2.6. The Generalized Diffusion Operator, $\widetilde{D}(k,s)$

As we have remarked, – see Eq. (6) – all that is interesting in the relaxation is controlled by $\widetilde{D}(k,s)$, the transform of $\mathbb{D}(\epsilon) \equiv P\{\mu\mathcal{G}(\epsilon)\mu\}$, the non-local diffusion operator. We begin by evaluating it to lowest order in ϵ, using our calculus to get the ϵ^2 contribution,

$$
\begin{aligned}
\mathbb{D}(\epsilon) &= \mathbb{D}_0 + \epsilon^2 \mathbb{D}_2 + \cdots \\
\mathbb{D}_2\,(k,s) &= P\{\mu\, G_1 Q\, \Sigma_0 G_1 \mu\} \\
&= \tfrac{1}{2\pi} \int dk' \Theta(k-k') P\{\mu\, G_1(k,s) \cdot\, QG(k',s) \cdot G_1(k,,s) \cdot \mu\} \\
&= \tfrac{1}{2\pi} \int dk' \Theta(k-k') Z(k,k',s).
\end{aligned}
\tag{34}
$$

The computation of Z is straightforward:

Write $\phi_1(\mu,k,s) = G_1(k,s)\cdot\mu = \dfrac{1}{d(k,\mu)}\left[\mu - \dfrac{\Lambda_1}{\Lambda}(k)\right]$, then

$$
\begin{aligned}
\phi_2(\mu,k,k',s) &= G(k',s)\cdot\phi_1(\mu,k,s) \\
&= \frac{1}{d(k',\mu)}\left[\phi_1(\mu,k) + \frac{1}{1-\Lambda(k')}\left\langle \frac{\phi_1(k)}{d(k')}\right\rangle\right],
\end{aligned}
$$

and finally

$$
\begin{aligned}
Z &= \phi_3(k,k',s) \\
&= P\left\{\mu G_1(k,s)\cdot Q\phi_2(\mu,k,k',s)\right\} \\
&= P\left\{\mu\, G_1(k,s)\cdot QG(k',s)\cdot\phi_1(\mu,k,s)\right\} \\
&= \left\langle \frac{\mu}{d(k)}\phi_2(k,k',s)\right\rangle - \frac{\Lambda_1}{\Lambda}(k)\left\langle \frac{1}{d(k)}\phi_2(k,k',s)\right\rangle.
\end{aligned}
\tag{35}
$$

After some algebra we find the surprisingly compact:

$$
Z = \phi_3(k,k',s) = \left\langle \frac{1}{d^2(k)\,d(k')}(\mu - \frac{\Lambda_1}{\Lambda}(k))^2\right\rangle + \tfrac{1}{1-\Lambda(k',s)}(\Delta_1 - \frac{\Lambda_1}{\Lambda}(k)\Delta)^2
\tag{36}
$$

with $\Delta(k,k',s) = \left\langle \frac{1}{d^2(k)\,d(k')}\right\rangle$ and $\Delta_1(k,k',s) = \left\langle \frac{\mu}{d^2(k)d(k')}\right\rangle$.

3. SOME CONSEQUENCES

3.1. The Markoffian Limit

The limit $s \to 0$, $k \to 0$ causes Λ_1 to vanish, converts Δ and Δ_1 to Λ and Λ_1 and, by virtue of Eq. (8b) causes Z to vanish. Thus, the diffusion co-efficient is unchanged to order ϵ^2. Apparently, small fluctuations in density do not alter the overall, "gross" co-efficient of diffusion. Restriction on the result may be seen by recalling

$$\lim \widetilde{D}(k,s) = \lim \frac{1}{2\pi}\int dk'\Theta(k-k')\ Z(k,k',s)$$

and noting that singular behavior of $\Theta(k')$ at $k'=0$ may spoil the conclusion. This occurs when the area under the auto-correlation curve is infinite, in the case when "uniform dichotomic or N-chotomic" fluctuations, for example (note Eq. (33)).

While the diffusion equation is unaltered, the fluctuations do affect the shape of an evolving burst. Consider the second spatial moment. The usual expansion in k^2 combined with Eq. (6) gives the effect of fluctuations upon the (transformed) moment as

$$\delta\langle\widetilde{x}^2(s)\rangle = \frac{2}{s^2}\int \frac{d\mathrm{k}'}{2\pi}\Theta(k')Z(0,k',s).$$

Since $Z=\phi_3$ vanishes as $s\to 0, \delta < x^2(t) >$ is o(t), and by virtue of the connection between $2d$ moment and diffusion coefficient we see again that the co-efficient is unaltered. Fluctuations affect the second moment modestly over all time scales.

If we are interested only in the Markoffian limit, the calculation described by Eqs. (34–36) may be simplified considerably for, throughout, "k" is simply a parameter and, along with "s," may be set to zero early in the calculation, thus,

$$\begin{aligned}
&\phi_1(\mu,0,0) = G_1(0,0)\cdot\mu = \mu,\\
&\phi_2(\mu,0,k',0) = G(k',0)\cdot\phi_1(0,k) = \frac{1}{d(k',\mu)}\left[\mu + \frac{\Lambda_1(k')}{1-\Lambda(k')}\right]\\
&Z(0,k',0) = \phi_3(0,k',0) = P\{\mu, G_1(0,0)\cdot Q\phi_2(\mu,0,k',0)\} = \langle\mu\phi_2\rangle\\
&\qquad\qquad = \Lambda_2 + \frac{\Lambda_1^2}{1-\Lambda} = 0. \quad \text{(again Eq. (8b).)}
\end{aligned}$$

Still simpler, after remarking that $\phi_1(\mu,0,0) = G_1(0,0)\cdot\mu = \mu,$ Eq. (35) and Eq. (25) imply at once the vanishing of $Z(0,k',0)$.

3.2. All Orders of ϵ

If we examine the ϵ-expansion, we see that this *last* argument for the vanishing appears to hold-almost by "easy inspection"– for all orders of ϵ in the expansion. In detail, every term in the expansion of $\mathcal{G}(\epsilon) =$ leads with $...\Sigma(\epsilon)G_1$, while every term in the expansion of $\Sigma(\epsilon)$ leads with $....Q\,G\theta\rangle_\alpha$. Thus the structure $Q\,G(k')G_1\cdot\mu$ is omnipresent,and vanishes in the Markoffian limit. One can accept this term-by-term argument or one can re-arrange the series, so that one faces a summed operator multiplying $QG(k',s)\;G_1(k,s)\mu$. The result, the vanishing, is the same.

4. THE DIFFUSION EQUATION WITH RANDOM COEFFICIENT

Another model one might consider is that of a diffusion equation whose co-efficient is space-dependent in a random manner. This model is completely macroscopic, and it will lead to a different conclusion.

Consider the system

$$\frac{\partial n}{\partial t}+\frac{\partial j}{\partial x}=0 \quad D(x,\alpha)\frac{\partial n}{\partial x}+j=0$$

in vector form, with $\psi=\binom{j}{n}$. Then

$$A\frac{\partial}{\partial t}\psi+A^T\psi+E\frac{\partial}{\partial \mathrm{x}}\psi+\epsilon\,\theta(x,\alpha)C\frac{\partial}{\partial \mathrm{x}}\psi=0$$

or

$$\mathcal{L}_0\psi+\epsilon\,\theta(x,\alpha)C\frac{\partial}{\partial \mathrm{x}}\psi=0.$$

The matrices

$E=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $C=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ are trivial, commuting with the projection (averaging-) operators, P_α and Q_α. Now, operating upon the equation with them, in turn, leads to

$$\mathcal{L}_0F(x,t)-\epsilon^2C\;P_\alpha\left\{\theta\frac{\partial}{\partial \mathrm{x}}\;\frac{1}{\mathcal{L}_0+\epsilon\,Q\,\theta\;C}\,\theta\right\}C\;\frac{\partial}{\partial \mathrm{x}}F(x,t)=0$$

with $F(x,t)=P_\alpha\,\psi\,(x,t,\alpha)$.

We will simply evaluate the alteration of the diffusion operator to lowest order. Thus we have the easy computation of

$$C\;P_\alpha\left\{\theta\frac{\partial}{\partial x}\;\frac{1}{\mathcal{L}_0}\theta\right\}C\;\frac{\partial}{\partial x}F(x,t).$$

With $F(x,t)=f(k,s)\exp[ikx+st]$ and $\theta(x,\alpha)=\int \frac{dk'}{2\pi}\widetilde{\theta}(k',\alpha)e^{ik'x}$ and

$$\frac{1}{\mathcal{L}_0(k,s)}=\frac{1}{s+k^2}\begin{pmatrix}-ik & s\\ 1 & -ik\end{pmatrix} \quad \text{and} \quad C\frac{1}{\mathcal{L}_0(k,s)}C=\frac{-ik}{s+k^2}C$$

we obtain

$$[E\mathcal{L}_0(k,\ s)-\epsilon^2\,ik\,\mathcal{K}(k,\ s)C]\widetilde{F}(k,s)=1.$$

with

$$\mathcal{K}(k,\ s)=\int\frac{dk'}{2\pi}\,\Theta(k')\,\frac{(k+k')^2}{s+(k+k')^2} \tag{37}$$

(Details of the calculation are presented in Appendix C)
Now, setting the corresponding determinant to zero,

$$\text{Det }[sA+A^T\ +ik\ E\ \ -\epsilon^2\,ik\,\mathcal{K}(k,\ s)C]\ =0$$

we find

$$s+\{1-\epsilon^2\ \mathcal{K}(k,\ \ s)\}k^2=0$$

and its Markoffian counterpart.

$$s+\left\{1-\epsilon^2\int\frac{dk'}{2\pi}\Theta(k')\right\}k^2=0$$

or

$$s+\{1-\epsilon^2<\theta^2>\}k^2\ \ =0,$$

In this case the diffusion co-efficient *is* altered. One notes, too, that when "constant statistics," i.e. very great correlation length, is assumed, so that $\Theta(k')\to 2\pi\theta^2\delta(k')$ appears in $\mathcal{K}(k,s)$, the subsequent Markoffian result will be ambiguous – it will depend upon how the limits $(s,k\to 0)$ are approached.

5. LAST COMMENTS

We have analyzed a non-trivial case in the transport of particles in a medium whose density of scatterers is stochastic. The particles scatter against host centers isotropically and with no loss of energy, the particle distribution evolving according to the traditional transport equation. It is perhaps no surprise that after some time the distribution approaches a solution to the macroscopic time-dependent diffusion equation. It is a surprise that the diffusion co-efficient associated with the equation is the "un-perturbed" or average diffusion co-efficient. There is no renormalization. This result is shown in some detail for the leading correction caused by stochasticity, and a concise argument shows that the result holds in higher order, term-by-term. (We have nothing much to say about the convergence of the series.) And we note that the result is "model-sensitive" – a less realistic model may lead to renormalization and ambiguity.

A final comment: one is asked about the effect that the addition of capture will have on the asymptotics, and the "no-renormalization" result. This is not simple because – at the very least – a new time scale enters the problem, and spatial relaxation in the limit is no longer characterized by $k \to 0$ but by $k \to k_0 = i/L$ where L is the dominant (macroscopic) relaxation length. Since L may depend upon the stochastic features, the analysis is not easy.

APPENDIX A

In the case of an uniform medium L is replaced by L_0. To evaluate the diffusion operator, call

$$\frac{1}{\frac{\partial}{\partial t} + Q\mathrm{L}_0} \mu \frac{\partial}{\partial x} F_0(x,t) = E(x,\mu,t), \text{ whence}$$

$$\frac{\partial E}{\partial t} + Q\left[\mu \frac{\partial E}{\partial x} + E(x,\mu,t)\right] = \mu \frac{\partial}{\partial x} F_0(x,t). \tag{A.1}$$

In terms of Fourier–Laplace transforms, this is

$$(1+s+ik\mu)\widetilde{E}(k,\mu,s) = P(1+ik\mu)\widetilde{E}(k,\mu,s) + ik\mu \widetilde{F}_0(k,s), \tag{A.2}$$

where we have chosen a particular solution $(E(t=0)=0)$.

If, then, we divide Eq. (A2) by $d=(1+s+ik\mu)$ and average (apply P) we are led after a little algebra, to Eq. (7).

The traditional treatment of Eq. (1), with uniform host and isotropic source yields

$$\widetilde{F}_0(k,s) = \frac{\Lambda(k,s)}{1-\Lambda(k,s)},$$

which implies $\widetilde{D}(k,s) = [1-(1+s)\Lambda(k,s)]\ \frac{1}{k^2\Lambda(k,s)}$.
This expression may be seen to be identical with Eq. (7) if we use Eq. (8).

APPENDIX B

Equation (3a) may be written

$$\left(\frac{\partial}{\partial t} + L_0\right) F(x,\mu,t,\alpha) + \epsilon\theta(x,\alpha) Q F(x,\mu,t,\alpha) = 0 \tag{B.1}$$

whence

$$\left(\frac{\partial}{\partial t} + L_0\right) P_\alpha F + \epsilon\ Q\ P_\alpha \theta(x,\alpha)\ Q_\alpha F = 0,$$

and

$$\left(\frac{\partial}{\partial t} + L_0\right) Q_\alpha F + \epsilon\ Q\ Q_\alpha \theta(x,\alpha)[\ P_\alpha F\ + Q_\alpha F] = 0.$$

Eliminating $Q_\alpha F$ gives

$$\left(\frac{\partial}{\partial t} + L_0\right) P_\alpha F - \epsilon^2\ Q\ \Sigma(\epsilon)\ Q\ P_\alpha F = 0, \tag{B.2}$$

with

$$\begin{aligned}\Sigma(\epsilon) &= P_\alpha \left[\theta(x,\alpha)\ \frac{1}{\frac{\partial}{\partial t} + \mathrm{L}_0 + \epsilon\ Q\ Q_\alpha \theta(x,\alpha)} \theta(x,\alpha)\right] \\ &= P_\alpha[\theta(x,\alpha)\ \sigma(\epsilon,\alpha)\theta(x,\alpha)],\end{aligned} \tag{B.3}$$

a complicated operator which acts on functions of (x,μ,t).

APPENDIX C

On Eq. 37 from

$$C\ P_\alpha \{\theta(x,\alpha)\frac{\partial}{\partial x}\ \frac{1}{\underset{\sim 0}{\mathcal{L}}}\theta(x,\alpha)\}\, C\ \frac{\partial}{\partial x}\ F(x,t). \tag{*}$$

to

$$[E\mathcal{L}_0(k,\ s) - \epsilon^2\, ik\,\mathcal{K}(k,\ s)C]\widetilde{F}(k,s) = 1.$$

where

$$\mathcal{K}(k,\ s) = \int \frac{dk'}{2\pi}\,\Theta(k')\,\frac{(k+k')^2}{s+(k+k')^2}. \tag{C.1}$$

Several steps are required. First, recall that $F(x,t)$ is a 2-component vector, and C is a trivial 2×2 matrix, $C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, a "projector," having the properties $C^2 = C$ and $C\begin{pmatrix} a & b \\ c & d \end{pmatrix}C = dC$. Also, $\frac{1}{\underset{\sim 0}{\mathcal{L}}}$ is a matrix of operators. We can use these properties to simplify (*), writing

$$(*)\ = P_\alpha\{\theta(x,\alpha)\frac{\partial}{\partial x}C\,\frac{1}{\underset{\sim 0}{\mathcal{L}}}C\theta(x,\alpha)\}\ \frac{\partial}{\partial x}F(x,t)$$

$$(*)\ =\ P_\alpha\{\theta(x,\alpha)\frac{\partial}{\partial x}\mathcal{L}_2 C\theta(x,\alpha)\}\ \frac{\partial}{\partial x}F(x,t).$$

with $\mathcal{L}_2$ no longer a matrix, but the (2,2) element of the inverse, $\frac{1}{\underset{\sim 0}{\mathcal{L}}}$.

To simplify further, we write

$$(*) = P_\alpha\{\theta(x,\alpha)\frac{\partial}{\partial \mathrm{x}}\mathcal{L}_2\, G(x,\alpha,t)\}. = P_\alpha\{\theta(x,\alpha)\, H(x,\alpha,t)\}$$

Next, since we are examining the Fourier–Laplace transform, that quantity is

$$\widetilde{(*)} = \frac{1}{2\pi}\int dk''\ \ P_\alpha\{\widetilde{\theta}(k'',\alpha)\ \ \widetilde{H}(k-k'',\alpha,\ s)\}.$$

Since $\widetilde{H}(k,\alpha,\ s) = ik\mathcal{L}_2(k,s)\,\widetilde{G}(k,\alpha,s)$ and

(The exponentials are eigenfunctions of $\mathcal{L}_2$, the eigenvalues being $\mathcal{L}_2(k,s) = \frac{-ik}{s+k^2}$.)

$$\widetilde{G}(k,\alpha,s) = \int dk'\,\widetilde{\theta}(k-k',\alpha)\, i\,k'\,C\widetilde{F}(k',s).$$

Assembling the pieces gives

$$\widetilde{(*)} = \frac{1}{2\pi}\int dk''\;\; P_\alpha\,\{\widetilde{\theta}(k'',\alpha) i(k-k'')\mathcal{L}_2(k-k'',s)\,\widetilde{G}(k-k'',\alpha,s)\}.$$

$$\widetilde{(*)} = \left(\frac{1}{2\pi}\right)^2 \int dk'' \int dk' i(k-k'')\mathcal{L}_2(k-k'',s) P_\alpha\,\{\widetilde{\theta}(k'',\alpha)\widetilde{\theta}(k-k''-k',\alpha)\} \cdot\{i\,k' C\widetilde{F}(k',s)\}$$

Since $P_\alpha\{\widetilde{\theta}(k',\alpha)\widetilde{\theta}(k'',\alpha)\} = 2\pi\,\Theta(k')\;\;\delta(k'+k^{'\prime})$ we have

$$\widetilde{(*)} = \frac{1}{2\pi}\int dk'' \int dk' i(k-k'')\mathcal{L}_2(k-k'',s)\Theta(k'')\,\delta(k-k')\;\; i\,k'\,C\widetilde{F}(k',s)$$

$$\widetilde{(*)} = \frac{1}{2\pi}\left\{\int dk''\,\Theta(k'')\;\; i(k-k'')\mathcal{L}_2(k-k'',s)\right\}\;\; i\,k\,C\widetilde{F}(k,s)$$

$$\widetilde{(*)} = \frac{1}{2\pi}\left\{\int dk''\,\Theta(k'')\, i(k-k'')\mathcal{L}_2(k-k'',s)\right\}\;\; i\,k\,C\widetilde{F}(k,s).$$

Using $\Theta(k'') = \Theta(-k'')$ and the expression for $\mathcal{L}_2$ brings us to the desired

$$\widetilde{(*)} = ik\,\mathcal{K}(k,\,s) C\widetilde{F}(k,s).$$

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